

On improvement of the concavity of convex measures

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Abstract

We prove that a general class of measures, which includes log-concave measures, are $\frac{1}{n}$ -concave in the terminology of Borell under additional assumptions on the measure or on the sets, such as symmetries. This generalizes results of Gardner and Zvavitch [8].

Keywords: Brunn-Minkowski inequality, convex measure, Gaussian measure

1 Introduction

The classical Brunn-Minkowski inequality asserts that for every Borel sets $A, B \subset \mathbb{R}^n$ and for every $\lambda \in [0, 1]$, one has

$$|(1 - \lambda)A + \lambda B|^{\frac{1}{n}} \geq (1 - \lambda)|A|^{\frac{1}{n}} + \lambda|B|^{\frac{1}{n}} \quad (1)$$

where $|\cdot|$ denotes the Lebesgue measure. The Brunn-Minkowski inequality is a beautiful and powerful inequality in geometry and in analysis which leads to many interesting consequences. For more informations about this inequality and its ramifications with several mathematical theories, see the survey of Gardner [7]. See also the book of Schneider [22], a reference in the convex Brunn-Minkowski theory.

Recently, Gardner and Zvavitch [8] proved that the Gaussian measure γ_n in \mathbb{R}^n , defined by

$$d\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}} dx, \quad x \in \mathbb{R}^n$$

where $|\cdot|$ denotes the Euclidean norm, satisfies a Brunn-Minkowski type inequality of the form (1) for particular sets. Namely they showed that for coordinate boxes A and B in \mathbb{R}^n , *i.e.* a product of intervals, containing the origin, or for $A, B \subset \mathbb{R}^n$ which are dilates of the same symmetric convex set, and for every $\lambda \in [0, 1]$, one has

$$\gamma_n((1-\lambda)A + \lambda B)^{\frac{1}{n}} \geq (1-\lambda)\gamma_n(A)^{\frac{1}{n}} + \lambda\gamma_n(B)^{\frac{1}{n}}, \quad (2)$$

and they conjectured that inequality (2) holds for every $A, B \subset \mathbb{R}^n$ convex symmetric.

As a consequence of the Prékopa-Leindler inequality [19], [15], [20], the Gaussian measure satisfies for every Borel sets $A, B \subset \mathbb{R}^n$ and for every $\lambda \in [0, 1]$,

$$\gamma_n((1-\lambda)A + \lambda B) \geq \gamma_n(A)^{1-\lambda} \gamma_n(B)^\lambda. \quad (3)$$

Using the terminology of Borell [2] (see Section 2 below for further details), this inequality means that the Gaussian measure is a log-concave measure. By inequality on means, inequality (2) is stronger than inequality (3), hence the results of Gardner and Zvavitch improves the concavity of the Gaussian measure by showing that this measure is $\frac{1}{n}$ -concave if restricted to a particular class of sets.

We will see in this paper that these results of Gardner and Zvavitch can be extended to a more general class of measures called *convex measures*, which includes log-concave ones and thus the Gaussian measure. This is the mathematical underlying idea of the Gaussian Brunn-Minkowski inequality (2), *i.e.* under symmetry assumptions, one can improve a certain property, here the concavity of a measure. However, we will see that symmetries are not the only hypothesis that permits to improve the concavity of a measure.

This paper is devoted to study the following question:

Question 1. For which $s \in [-\infty, +\infty]$, for which class \mathcal{M} of non-negative measures in \mathbb{R}^n and for which class \mathcal{C} of couples of Borel subsets of \mathbb{R}^n one has for every $\mu \in \mathcal{M}$, for every $(A, B) \in \mathcal{C}$ such that $\mu(A)\mu(B) > 0$ and for every $\lambda \in [0, 1]$,

$$\mu((1-\lambda)A + \lambda B) \geq ((1-\lambda)\mu(A)^s + \lambda\mu(B)^s)^{\frac{1}{s}} \quad ? \quad (4)$$

The right-hand side of inequality (4) has to be interpreted by $\mu(A)^{1-\lambda}\mu(B)^\lambda$ for $s = 0$, by $\min(\mu(A), \mu(B))$ for $s = -\infty$ and by $\max(\mu(A), \mu(B))$ for $s = +\infty$.

Borell [2] (see also [3]) proved that Question 1. has a positive answer if \mathcal{M} is the class of s -concave measures in \mathbb{R}^n , $s \in [-\infty, +\infty]$, and if \mathcal{C} is the class of couples of Borel subsets of \mathbb{R}^n (see Section 2).

If restricted to the Lebesgue measure, Question 1. has been explored for $s = 1$, by Bonnesen [1], and is still studied (see *e.g.* [11]).

The main results of this paper are contained in the following theorem:

Theorem 1.

1. Let μ be an unconditional log-concave measure in \mathbb{R}^n and let A be an unconditional convex subset of \mathbb{R}^n . Then, for every $A_1, A_2 \in \{\alpha A; \alpha > 0\}$ and for every $\lambda \in [0, 1]$, we get

$$\mu((1 - \lambda)A_1 + \lambda A_2)^{\frac{1}{n}} \geq (1 - \lambda)\mu(A_1)^{\frac{1}{n}} + \lambda\mu(A_2)^{\frac{1}{n}}.$$

2. Let μ_i , $1 \leq i \leq n$, be measures with densities $\phi_i : \mathbb{R} \rightarrow \mathbb{R}_+$ such that ϕ_i are non-decreasing on $(-\infty; 0]$ and non-increasing on $[0; +\infty)$. Let μ be the product measure of μ_1, \dots, μ_n and let $A, B \subset \mathbb{R}^n$ be the product of n Borel subsets of \mathbb{R} such that $0 \in A \cap B$. Then, for every $\lambda \in [0, 1]$, we get

$$\mu((1 - \lambda)A + \lambda B)^{\frac{1}{n}} \geq (1 - \lambda)\mu(A)^{\frac{1}{n}} + \lambda\mu(B)^{\frac{1}{n}}.$$

In the next section, we introduce some terminologies that will be needed. The third section is devoted to prove Theorem 1. In the last section, we discuss how these results improve concavity properties of the (extended) parallel volume.

2 Preliminaries

We work in the Euclidean space \mathbb{R}^n , $n \geq 1$, equipped with the ℓ_2^n norm $|\cdot|$, whose closed unit ball is denoted by B_2^n , the unit sphere by \mathcal{S}^{n-1} and the canonical basis by $\{e_1, \dots, e_n\}$. We also denote by $|\cdot|$ the Lebesgue measure in \mathbb{R}^n . For $u \in \mathcal{S}^{n-1}$, we denote by u^\perp the hyperplane orthogonal to u . For non-empty sets A, B in \mathbb{R}^n we define their *Minkowski sum*

$$A + B = \{a + b; a \in A, b \in B\}.$$

A subset A of \mathbb{R}^n is said to be *symmetric* if $A = -A$. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *unconditional* if there exists a basis (a_1, \dots, a_n) of \mathbb{R}^n such that for every $x = \sum_{i=1}^n x_i a_i \in \mathbb{R}^n$ and for every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$, one has $f(\sum_{i=1}^n \varepsilon_i x_i a_i) = f(x)$. A subset A of \mathbb{R}^n is said to be *unconditional* if the indicator function of A , denoted by 1_A , is unconditional. A (non-negative) measure with density is said to be *symmetric* (resp. *unconditional*) if its density is even (resp. unconditional).

Let us recall some terminologies and results about s -concave measures introduced by Borell in [2]. One says that a measure μ in \mathbb{R}^n is s -concave, $s \in [-\infty, +\infty]$, if the inequality

$$\mu((1-\lambda)A + \lambda B) \geq ((1-\lambda)\mu(A)^s + \lambda\mu(B)^s)^{\frac{1}{s}} \quad (5)$$

holds for every Borel subsets $A, B \subset \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$ and for every $\lambda \in [0, 1]$. The limit cases are interpreted by continuity, as mentioned in the introduction. The 0-concave measures are also called *log-concave measures*.

Notice that a s -concave measure is r -concave for every $r \leq s$. Thus, every s -concave measure is $-\infty$ -concave. The $-\infty$ -concave measures are also called *convex measures*.

From inequality (3), the Gaussian measure is a log-concave measure and as a consequence of the Brunn-Minkowski inequality (1), the Lebesgue measure is a $\frac{1}{n}$ -concave measure.

For every $s \in [-\infty, +\infty]$, Borell gave a complete description of s -concave measures. In particular for $s \leq \frac{1}{n}$, Borell showed that every measure μ absolutely continuous with respect to the n -dimensional Lebesgue measure is s -concave if and only if its density is a γ -concave function, with

$$\gamma = \frac{s}{1 - sn} \in [-\frac{1}{n}, +\infty],$$

where a function $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is said to be γ -concave, with $\gamma \in [-\infty, +\infty]$, if the inequality

$$f((1-\lambda)x + \lambda y) \geq ((1-\lambda)f(x)^\gamma + \lambda f(y)^\gamma)^{\frac{1}{\gamma}}$$

holds for every $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$ and for every $\lambda \in [0, 1]$. As for the s -concave measures, the limit cases are interpreted by continuity. Notice that a 1-concave function is concave on its support, that f is a $-\infty$ -concave function if and only if f has its level sets $\{x; f(x) \geq t\}$ convex, and that f is a $+\infty$ -concave function if and only if f is constant on its support.

A natural generalization of convex measures are measures with $-\infty$ -concave density. From the results of Borell, a measure with γ -concave density where $\gamma < -\frac{1}{n}$, does not satisfied a concavity property of the form (5) (but satisfies another form of concavity [5]). However, we will show that if restricted to particular sets, such measures are $\frac{1}{n}$ -concave.

In this paper, we call *sub-convex measure* a measure with $-\infty$ -concave density. Notice that convex measures are sub-convex.

3 Brunn-Minkowski type inequality for sub-convex measures

In this section, we partially answer to Question 1. by investigating possible improvements of the concavity of sub-convex measures. Gardner and Zvavitch [8] noticed in the case of the Gaussian measure, that the position of the sets A and B plays an important role. Indeed, since for s -concave probability measures μ , with $s \leq 0$, the density tends to 0 at infinity and the support of the density can be equal to \mathbb{R}^n , one can find sets A and B such that A contain the origin and $\frac{A+B}{2}$ is far from the origin. Thus for $r > 0$, the inequality

$$\mu\left(\frac{A+B}{2}\right)^r \geq \frac{\mu(A)^r + \mu(B)^r}{2}$$

will not be satisfied. Hence, the position of the sets A and B is an inherent constraint of the problem. Notice also that in the definition of s -concave measures, the condition $\mu(A)\mu(B) > 0$ is already a constraint on the position of A and B with respect to the support of μ .

Notice that Question 1. has a positive answer for $s = +\infty$ if \mathcal{M} is the class of convex measures and if \mathcal{C} is the class of couples of Borel sets with same measure. Indeed, one then has for every $\lambda \in [0, 1]$

$$\mu((1-\lambda)A + \lambda B) \geq \inf(\mu(A), \mu(B)),$$

by definition. Since $\mu(A) = \mu(B)$, it follows that

$$\mu((1-\lambda)A + \lambda B) \geq \mu(A) = \max(\mu(A), \mu(A)) = \max(\mu(A), \mu(B)).$$

Notice also that for every measure μ and for every Borel sets A, B such that $A \subset B$, one has for every $\lambda \in [0, 1]$,

$$\mu((1-\lambda)A + \lambda B) \geq \min(\mu(A), \mu(B)),$$

since in this case one has, $(1-\lambda)A + \lambda B \supset (1-\lambda)A + \lambda A \supset A$.

3.1 The case of symmetric measures and symmetric sets

Under symmetry assumptions, the best concavity one can obtain is $\frac{1}{n}$ by considering for example the Lebesgue measure, which fulfils a lot of symmetries (unconditional), and two dilates of B_2^n (which are unconditional). This was noticed by Gardner and Zvavitch [8] also for the Gaussian measure.

A sufficient condition to ensure that a measure μ in \mathbb{R}^n is $\frac{1}{n}$ -concave in the class of dilates of a fixed Borel set $A \subset \mathbb{R}^n$ is that the function $t \mapsto \mu(tA)$ is $\frac{1}{n}$ -concave. The following proposition gives a sufficient condition for this.

Proposition 3.1. *Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a measurable function such that for every $x \in \mathbb{R}^n$, the function $t \mapsto \phi(tx)$ is non-increasing on \mathbb{R}_+ . Let μ be a measure with density ϕ and A be a Borel subset of \mathbb{R}^n containing 0. If the function $t \mapsto \mu(e^t A)$ is log-concave on \mathbb{R} , then the function $t \mapsto \mu(tA)$ is $\frac{1}{n}$ -concave on \mathbb{R}_+ .*

Proof. Let μ be a measure with density ϕ satisfying the assumptions of Proposition 3.1 and let A be a Borel subset of \mathbb{R}^n containing 0. Let us denote $F(t) = \mu(tA)$, for $t \in \mathbb{R}_+$. Notice that F is non-decreasing and continuous on \mathbb{R}_+ . By assumption, the function $t \mapsto F(e^t)$ is log-concave on \mathbb{R} . It follows that the right derivative of F , denoted by F'_+ , exists everywhere and that $t \mapsto tF'_+(t)/F(t)$ is non-increasing on $(0, +\infty)$.

Notice that the function F is $\frac{1}{n}$ -concave on \mathbb{R}_+ if and only if the function

$$t \mapsto \frac{tF'_+(t)}{F(t)} \frac{F(t)^{\frac{1}{n}}}{t}$$

is non-increasing on \mathbb{R}_+ . A direct change of variables shows that

$$\frac{F(t)}{t^n} = \int_A \phi(tx) dx.$$

By assumption, the function $t \mapsto \phi(tx)$ is non-increasing on \mathbb{R}_+ . It follows that the function $t \mapsto F(t)^{\frac{1}{n}}/t$ is non-increasing on $(0, +\infty)$. Hence, the function $t \mapsto (F(t)^{\frac{1}{n}})'_+$ is non-increasing on $(0, +\infty)$ as the product of two non-negative non-decreasing functions on $(0, +\infty)$. We conclude that F is $\frac{1}{n}$ -concave on \mathbb{R}_+ . \square

Remarks.

1. Proposition 3.1 is established in [8] for the Gaussian measure by differentiating twice.
2. The assumption $t \mapsto \phi(tx)$ is non-increasing on \mathbb{R}_+ is satisfied if ϕ is an even $-\infty$ -concave function.

Proposition 3.1 is related to the (B)-conjecture. This conjecture was posed by W. Banaszczyk [14] and asks whether the function $t \mapsto \gamma_n(e^t A)$ is log-concave on \mathbb{R} , for every convex symmetric set $A \subset \mathbb{R}^n$. The (B)-conjecture was proved by Cordero-Erausquin, Fradelizi and Maurey in [4]. In the same paper [4], the authors have also showed that for every unconditional log-concave measure μ in \mathbb{R}^n and for every unconditional convex subset $A \subset \mathbb{R}^n$, the function $t \mapsto \mu(e^t A)$ is log-concave on \mathbb{R} . Using this and the point 2. of the previous remark, we may apply Proposition 3.1 to get the following corollary:

Corollary 3.2. *Let μ be an unconditional log-concave measure in \mathbb{R}^n and let A be an unconditional convex subset of \mathbb{R}^n . Then, the measure μ is $\frac{1}{n}$ -concave in the class of dilates of A . More precisely, for every $A_1, A_2 \in \{\alpha A; \alpha > 0\}$ and for every $\lambda \in [0, 1]$, we get*

$$\mu((1 - \lambda)A_1 + \lambda A_2)^{\frac{1}{n}} \geq (1 - \lambda)\mu(A_1)^{\frac{1}{n}} + \lambda\mu(A_2)^{\frac{1}{n}}.$$

Remark. Very recently, Livne Bar-on [16] and Saroglou [21] proved, using different methods, that in dimension 2 for the uniform measure μ_K on a symmetric convex set $K \subset \mathbb{R}^2$, the function $t \mapsto \mu_K(e^t A)$ is log-concave on \mathbb{R} for every symmetric convex set $A \subset \mathbb{R}^2$. However, for our problem, this information is not useful since the uniform measure on a convex subset of \mathbb{R}^n is a $\frac{1}{n}$ -concave measure.

A natural question is to ask if the role of the symmetry can be relaxed. If restricted to the Gaussian measure, it has been shown by Nayar and Tkocz in [18], that in dimension 2 there exists non-symmetric convex sets A and B in \mathbb{R}^2 satisfying $0 \in A \subset B$ and

$$\gamma_2\left(\frac{A+B}{2}\right)^{\frac{1}{2}} < \frac{\gamma_2(A)^{\frac{1}{2}} + \gamma_2(B)^{\frac{1}{2}}}{2}. \quad (6)$$

It is then direct to construct explicit counter-example in every dimension $n \geq 2$. Moreover, the counterexample in [18] shows more than inequality (6). It shows that

$$\gamma_2\left(\frac{A+B}{2}\right)^s < \frac{\gamma_2(A)^s + \gamma_2(B)^s}{2}, \quad (7)$$

for every $s \geq 1 - \frac{2}{\pi}$. However, it could be interesting to know what happens for $s \in (0, 1 - \frac{2}{\pi})$.

Notice that the same counterexample with the following log-concave unconditional measure instead of the Gaussian measure

$$d\mu(x, y) = e^{-|x|}e^{-|y|} dx dy, \quad (x, y) \in \mathbb{R}^2$$

satisfies inequality (7) for every $s > 0$.

Thus, in general, the symmetry assumption on the measure is not sufficient.

On the other hand, the concavity of a non-symmetric convex measure cannot be improved in general in the class of symmetric sets even in dimension 1:

Proposition 3.3. *Let $0 < s < 1$ and $r > s$. There exists a s -concave measure μ in \mathbb{R} and symmetric sets $A, B \subset \mathbb{R}$ such that*

$$\mu\left(\frac{A+B}{2}\right) < \left(\frac{\mu(A)^r + \mu(B)^r}{2}\right)^{\frac{1}{r}}.$$

Proof. Let us define $d\mu(x) = x^{1/\gamma} 1_{\mathbb{R}_+}(x) dx$, with $\gamma = \frac{s}{1-s} > 0$. Let us consider the sets $A = [-a, a]$ and $B = [-b, b]$ with $0 < a < b$. Notice that

$$\lim_{a \rightarrow 0} \mu\left(\frac{A+B}{2}\right) = \mu\left(\frac{B}{2}\right) = \frac{\mu(B)}{2^{\frac{1}{s}}} = \lim_{a \rightarrow 0} \left(\frac{\mu(A)^s + \mu(B)^s}{2}\right)^{\frac{1}{s}}.$$

Since $\mu(A) \neq \mu(B)$, it follows from inequalities on means that

$$\left(\frac{\mu(A)^s + \mu(B)^s}{2}\right)^{\frac{1}{s}} < \left(\frac{\mu(A)^r + \mu(B)^r}{2}\right)^{\frac{1}{r}}.$$

we conclude that for a sufficiently small,

$$\mu\left(\frac{A+B}{2}\right) < \left(\frac{\mu(A)^r + \mu(B)^r}{2}\right)^{\frac{1}{r}}.$$

□

Thus, in general, the symmetry assumption on the sets is not sufficient.

3.2 The case of sets with a maximal section of equal measure

In this section, we consider \mathcal{C} to be the class of couples of Borel subsets of \mathbb{R}^n having a maximal section of equal measure. A famous result of Bonnesen [1] (for convex sets) states that if $A, B \subset \mathbb{R}^n$ satisfy

$$\sup_{t \in \mathbb{R}} |A \cap (u^\perp + tu)|_{n-1} = \sup_{t \in \mathbb{R}} |B \cap (u^\perp + tu)|_{n-1},$$

where $|\cdot|_{n-1}$ denotes the $(n-1)$ -dimensional Lebesgue measure, then for every $\lambda \in [0, 1]$, one has

$$|(1-\lambda)A + \lambda B| \geq (1-\lambda)|A| + \lambda|B|.$$

There exists a functional version of Bonnesen's result established by Henstock and Macbeath [10] in dimension 1 (see Proposition 3.4 below) and later on by Dancs and Uhrin [5] in higher dimension (see Proposition 3.8 below).

Proposition 3.4 (Henstock, Macbeath [10]). *Let $\lambda \in [0, 1]$. Let $f, g, h : \mathbb{R} \rightarrow \mathbb{R}_+$ be non-negative measurable functions such that $\max(f) = \max(g)$ and such that for every $x, y \in \mathbb{R}$*

$$h((1-\lambda)x + \lambda y) \geq \min(f(x), g(y)).$$

Then, one has

$$\int_{\mathbb{R}} h(x) dx \geq (1-\lambda) \int_{\mathbb{R}} f(x) dx + \lambda \int_{\mathbb{R}} g(x) dx.$$

We deduce the following result:

Proposition 3.5. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$ be a $-\infty$ -concave function such that $\max(\phi)$ is attained in $a \in \mathbb{R}$. Let μ be a measure with density ϕ . Let A, B be Borel subsets of \mathbb{R} such that $a \in A \cap B$. Then, for every $\lambda \in [0, 1]$, we have*

$$\mu((1 - \lambda)A + \lambda B) \geq (1 - \lambda)\mu(A) + \lambda\mu(B).$$

Proof. Let $\lambda \in [0, 1]$. We define, for every $x \in \mathbb{R}$, $h(x) = \phi(x)1_{(1-\lambda)A+\lambda B}(x)$, $f(x) = \phi(x)1_A(x)$, $g(x) = \phi(x)1_B(x)$. Notice that for every $x, y \in \mathbb{R}$ one has

$$h((1 - \lambda)x + \lambda y) \geq \min(f(x), g(y)),$$

and $\max(f) = \max(g) = \phi(a)$. It follows from Proposition 3.4 that

$$\int_{\mathbb{R}} h(x) dx \geq (1 - \lambda) \int_{\mathbb{R}} f(x) dx + \lambda \int_{\mathbb{R}} g(x) dx.$$

In other words, we get

$$\mu((1 - \lambda)A + \lambda B) \geq (1 - \lambda)\mu(A) + \lambda\mu(B).$$

□

Remark. Proposition 3.5 was established in [8] in the particular case where μ is the Gaussian measure in \mathbb{R} and where $A, B \subset \mathbb{R}$ are convex. In the same paper, the authors were able to remove the convexity assumption for only one set, by using long computations and they did not know whether one can remove the convexity assumption on the second set. Our method bypass the use of geometric tools and relies on the functional version Proposition 3.4.

Conversely, if a measure μ in \mathbb{R} , admitting a density ϕ with respect to the Lebesgue measure, satisfies

$$\mu((1 - \lambda)A + \lambda B) \geq (1 - \lambda)\mu(A) + \lambda\mu(B),$$

for every $\lambda \in [0, 1]$ and for every symmetric convex sets $A, B \subset \mathbb{R}$, then one has for every $\lambda \in [0, 1]$ and for every $a, b \in \mathbb{R}_+$,

$$\int_{-(1-\lambda)a+\lambda b}^{(1-\lambda)a+\lambda b} \phi(x) dx \geq (1 - \lambda) \int_{-a}^a \phi(x) dx + \lambda \int_{-b}^b \phi(x) dx.$$

It follows that the function $t \mapsto \phi(t) + \phi(-t)$ is non-increasing on \mathbb{R}_+ . Notice that this condition is satisfied for more general functions than $-\infty$ -concave functions attaining the maximum at 0.

However, one can see with the same argument that if one assume $A, B \subset \mathbb{R}$ convex containing 0 (not necessarily symmetric), then it follows that the

density ϕ is necessarily non-decreasing on $(-\infty; 0]$ and non-increasing on $[0; +\infty)$. Notice that this is equivalent to the fact that the density ϕ is $-\infty$ -concave and $\max(\phi)$ is attained at 0.

By tensorization, Proposition 3.5 leads to the following corollary:

Corollary 3.6. *Let μ_i , $1 \leq i \leq n$, be measures with densities $\phi_i : \mathbb{R} \rightarrow \mathbb{R}_+$ such that ϕ_i are non-decreasing on $(-\infty; 0]$ and non-increasing on $[0; +\infty)$. Let μ be the product measure of μ_1, \dots, μ_n and let $A, B \subset \mathbb{R}^n$ be the product of n Borel subsets of \mathbb{R} such that $0 \in A \cap B$. Then, for every $\lambda \in [0, 1]$, we have*

$$\mu((1 - \lambda)A + \lambda B)^{\frac{1}{n}} \geq (1 - \lambda)\mu(A)^{\frac{1}{n}} + \lambda\mu(B)^{\frac{1}{n}}.$$

Proof. We follow [8]. By assumption, $A = \Pi_{i=1}^n A_i$ and $B = \Pi_{i=1}^n B_i$, where for every $i \in \{1, \dots, n\}$, A_i and B_i are Borel subsets of \mathbb{R} containing 0. Let $\lambda \in [0, 1]$. Notice that

$$(1 - \lambda)A + \lambda B = \Pi_{i=1}^n ((1 - \lambda)A_i + \lambda B_i).$$

Using Proposition 3.5 and an inequality of Minkowski (see e.g. [9]), one deduces that

$$\begin{aligned} \mu((1 - \lambda)A + \lambda B)^{\frac{1}{n}} &= (\Pi_{i=1}^n \mu_i((1 - \lambda)A_i + \lambda B_i))^{\frac{1}{n}} \\ &\geq (\Pi_{i=1}^n ((1 - \lambda)\mu_i(A_i) + \lambda\mu_i(B_i)))^{\frac{1}{n}} \\ &\geq (\Pi_{i=1}^n (1 - \lambda)\mu_i(A_i))^{\frac{1}{n}} + (\Pi_{i=1}^n \lambda\mu_i(B_i))^{\frac{1}{n}} \\ &= (1 - \lambda)\mu(A)^{\frac{1}{n}} + \lambda\mu(B)^{\frac{1}{n}}. \end{aligned}$$

□

Another consequence of Proposition 3.5 is that some particular product measures are concave measures if one set is a union of slabs containing the origin.

Corollary 3.7. *Let μ_1 be a measure with density $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$, such that ϕ is non-decreasing on $(-\infty; 0]$ and non-increasing on $[0; +\infty)$. Let μ_2 be a $(n - 1)$ -dimensional measure and let μ be the product measure of μ_1 and μ_2 . Let $A = A_1 \times \mathbb{R}^{n-1}$, where A_1 is a Borel subset of \mathbb{R} and let B be a Borel subset of \mathbb{R}^n such that $0 \in A \cap B$. Then, for every $\lambda \in [0, 1]$, we have*

$$\mu((1 - \lambda)A + \lambda B) \geq (1 - \lambda)\mu(A) + \lambda\mu(B).$$

Corollary 3.7 was established in [8] in the particular case where μ is the Gaussian measure and where one set is convex and with the weaker conclusion that the measure is $\frac{1}{n}$ -concave.

Proof. We follow [8]. Let us denote $B_S = P_{e_1}(B) \times \mathbb{R}^{n-1}$, where $P_{e_1}(B)$ denotes the orthogonal projection of B on the first coordinate axis. Then, for every $\lambda \in [0, 1)$, one has

$$(1 - \lambda)A + \lambda B = (1 - \lambda)A + \lambda B_S.$$

It follows, using Proposition 3.5, that

$$\begin{aligned} \mu((1 - \lambda)A + \lambda B) &= \mu((1 - \lambda)A + \lambda B_S) \\ &= \mu(((1 - \lambda)A_1 + \lambda P_{e_1}(B)) \times \mathbb{R}^{n-1}) \\ &= \mu_1((1 - \lambda)A_1 + \lambda P_{e_1}(B)) \cdot \mu_2(\mathbb{R}^{n-1}) \\ &\geq ((1 - \lambda)\mu_1(A_1) + \lambda\mu_1(P_{e_1}(B))) \cdot \mu_2(\mathbb{R}^{n-1}) \\ &= (1 - \lambda)\mu(A) + \lambda\mu(B_S) \\ &\geq (1 - \lambda)\mu(A) + \lambda\mu(B). \end{aligned}$$

□

In another hand, Proposition 3.4 can be turned in dimension n . First, let us define for a non-negative measurable function $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ and for $u \in \mathcal{S}^{n-1}$,

$$m_u(f) = \sup_{t \in \mathbb{R}} \int_{u^\perp} f(x + tu) \, dx.$$

Proposition 3.8 (Dancs, Uhrin [5]). *Let $-\frac{1}{n-1} \leq \gamma \leq +\infty$, $\lambda \in [0, 1]$ and $f, g, h : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be non-negative measurable functions such that for every $x, y \in \mathbb{R}^n$,*

$$h((1 - \lambda)x + \lambda y) \geq ((1 - \lambda)f(x)^\gamma + \lambda g(y)^\gamma)^{\frac{1}{\gamma}}.$$

If there exists $u \in \mathcal{S}^{n-1}$ such that $m_u(f) = m_u(g) < +\infty$, then

$$\int_{\mathbb{R}^n} h(x) \, dx \geq (1 - \lambda) \int_{\mathbb{R}^n} f(x) \, dx + \lambda \int_{\mathbb{R}^n} g(x) \, dx.$$

We deduce the following result. First, let us denote for a measure μ with density ϕ , for a Borel subset $A \subset \mathbb{R}^n$ and for a hyperplane $H \subset \mathbb{R}^n$,

$$\mu_{n-1}(A \cap H) = \int_{A \cap H} \phi(x) \, dx.$$

Proposition 3.9. *Let μ be a measure with density $\phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that ϕ is $-\frac{1}{n-1}$ -concave. Let A, B be Borel subsets of \mathbb{R}^n . If there exists $u \in \mathcal{S}^{n-1}$ such that*

$$\sup_{t \in \mathbb{R}} \mu_{n-1}(A \cap (u^\perp + tu)) = \sup_{t \in \mathbb{R}} \mu_{n-1}(B \cap (u^\perp + tu)),$$

then, for every $\lambda \in [0, 1]$, we have

$$\mu((1 - \lambda)A + \lambda B) \geq (1 - \lambda)\mu(A) + \lambda\mu(B).$$

Proof. Let $\lambda \in [0, 1]$. Let us take $f = \phi 1_A$, $g = \phi 1_B$ and $h = \phi 1_{(1-\lambda)A + \lambda B}$. Then, for every $x, y \in \mathbb{R}^n$, one has

$$h((1-\lambda)x + \lambda y) \geq ((1-\lambda)f(x)^\gamma + \lambda g(y)^\gamma)^\frac{1}{\gamma},$$

where $\gamma = -\frac{1}{n-1}$. Moreover,

$$\int_{u^\perp} f(x + tu) \, dx = \int_{A \cap (u^\perp + tu)} \phi(x) \, dx = \mu_{n-1}(A \cap (u^\perp + tu)).$$

It follows that $m_u(f) = m_u(g)$. From Proposition 3.8, we get that

$$\mu((1-\lambda)A + \lambda B) \geq (1-\lambda)\mu(A) + \lambda\mu(B).$$

□

4 Application to the parallel volume

Let us see how improvements of the concavity of sub-convex measures can improve the concavity of a generalized form of the parallel volume. The parallel volume of a Borel subset A of \mathbb{R}^n , namely the function $t \mapsto |A + tB_2^n|$, is a particularly interesting functional in geometry, which has been highlighted by the precursor works of Steiner in [23]. Even nowadays, the parallel volume and its generalized forms are still studied (see *e.g.* [12], [13]). Moreover, this notion of parallel volume leads to the powerful theory of mixed volumes (see [22] for further details).

As a consequence of the Brunn-Minkowski inequality (1), one can see that if $A \subset \mathbb{R}^n$ is convex, then the parallel volume of A is $\frac{1}{n}$ -concave on \mathbb{R}_+ . More generally, if a measure μ is s -concave, with $s \in [-\infty; +\infty]$, in the class of sets of the form $\{A + tB; t \in \mathbb{R}_+\}$, where A and B are convex subsets of \mathbb{R}^n , then the generalized parallel volume $t \mapsto \mu(A + tB)$ is s -concave on \mathbb{R}_+ . Indeed, for every $t_1, t_2 \in \mathbb{R}_+$ and for every $\lambda \in [0, 1]$, one gets

$$\begin{aligned} \mu(A + ((1-\lambda)t_1 + \lambda t_2)B) &= \mu((1-\lambda)(A + t_1B) + \lambda(A + t_2B)) \\ &\geq ((1-\lambda)\mu(A + t_1B)^s + \lambda\mu(A + t_2B)^s)^\frac{1}{s}. \end{aligned}$$

Using this and Corollary 3.6, we get the following corollary:

Corollary 4.1. *Let μ_i , $1 \leq i \leq n$, be measures with densities $\phi_i : \mathbb{R} \rightarrow \mathbb{R}_+$ such that ϕ_i are non-decreasing on $(-\infty; 0]$ and non-increasing on $[0; +\infty)$. Let μ be the product measure of μ_1, \dots, μ_n and let $A, B \subset \mathbb{R}^n$ be coordinate boxes containing the origin. Then the function $t \mapsto \mu(A + tB)$ is $\frac{1}{n}$ -concave on \mathbb{R}_+ .*

In the case of non-convex sets, this property of concavity is no more true in general, even for the classical parallel volume $|A + tB_2^n|$. However, some conditions are given on A in [6] for which the parallel volume of A is $\frac{1}{n}$ -concave on \mathbb{R}_+ . Notice that other concavity properties of generalized forms of the classical parallel volume have been established in [17].

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